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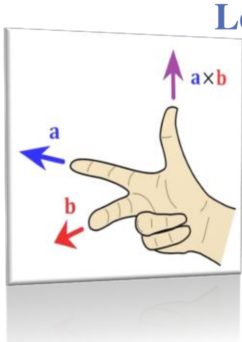
General Physics I

Mechanics: Principles & Applications

Lecture (3): Units, Dimensions and Vectors



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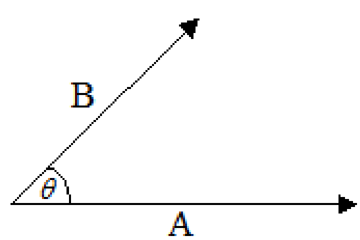
Product of vectors

There are two kinds of vector product the first one is called scalar product or dot product because the result of the product is a scalar quantity. The second is called vector product or cross product because the result is a vector perpendicular to the plane of the two vectors.



The scalar product

يعرف الضرب القياسي scalar product بالضرب النقطي dot product وتكون نتيجة الضرب القياسي لمتجهين كمية قياسية، وتكون هذه القيمة موجبة إذا كانت الزاوية المحصورة بين المتجهين بين 0 و 90 درجة وتكون النتيجة سالبة إذا كانت الزاوية المحصورة بين المتجهين بين 90 و 180 درجة وتساوي صفراً إذا كانت الزاوية 90.

A.B = +ve when $0 \le \theta < 90^o$
A.B = -ve when $90^o < \theta \le 180^o$
A.B = zero when $\theta = 90$

يعرف الضرب القياسي لمتجهين بحاصل ضرب مقدار المتجه الأول في مقدار المتجه الثاني في جيب تمام الزاوية المحصورة بينهما.

$$\vec{A}.\vec{B} = |A||B|\cos\theta$$

يمكن إيجاد قيمة الضرب القياسي لمتجهين باستخدام مركبات كل متجه كما يلي:

$$\vec{A} = A_x i + A_y j + A_z k$$
$$\vec{B} = B_x i + B_y j + B_z k$$

The scalar product is

$$\vec{A}.\vec{B} = (A_x i + A_y j + A_z k).(B_x i + B_y j + B_z k)$$

بضرب مركبات المتجه A في مركبات المتجه B ينتج التالي:

$$\vec{A}.\vec{B} = (A_x i.B_x i + A_x i.B_y j + A_x i.B_z k$$
$$+ A_y j.B_x i + A_y j.B_y j + A_y j.B_z k$$
$$+ A_z k.B_x i + A_z k.B_y j + A_z k.B_z k)$$

Therefore

$$\therefore \vec{A}.\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

The angle between the two vectors is

$$\cos\theta = \frac{\vec{A}.\vec{B}}{|A||B|} = \frac{A_x B_x + A_y B_y + A_z B_z}{|A||B|}$$

Example 1

Find the angle between the two vectors

$$\vec{A} = 2i + 3j + 4k \quad \vec{B} = i - 2j + 3k$$

Solution

$$\cos\theta = \frac{A_x B_x + A_y B_y + A_z B_z}{|A||B|}$$

$$A_x B_x + A_y B_y + A_z B_z = (2)(1)+(3)(-2)+(4)(3)=8$$

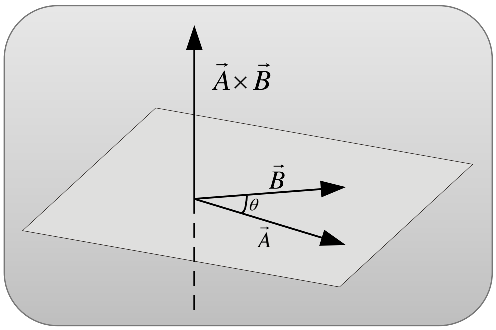
$$|A| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

$$|B| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$$

$$\cos\theta = \frac{8}{\sqrt{29}\sqrt{14}} = 0.397 \Rightarrow \theta = 66.6^o$$

The vector product

يعرف الضرب الاتجاهي *vector product* بـ *cross product* وتكون نتيجة الضرب الاتجاهي لمتجهين كمية متجهة. قيمة هذا المتجه $\vec{C}=\vec{A}\times\vec{B}$ واتجاهه عمودي على كل من المتجهين **A** و **B** وفي اتجاه دوران بريمة من المتجه **A** إلى المتجه **B** كما في الشكل التالي:



$$\vec{A}\times\vec{B}=AB\sin\theta$$

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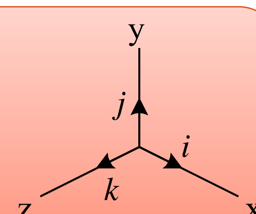
$$\vec{A}\times\vec{B}=\left(A_x i+A_y j+A_z k\right)\times\left(B_x i+B_y j+B_z k\right)$$

To evaluate this product we use the fact that the angle between the unit vectors *i*, *j* , *k* is 90°.

$$i\times i=0 \qquad i\times j=k \qquad i\times k=-j$$

$$j\times j=0 \qquad j\times k=i \qquad j\times i=-k$$

$$k\times k=0 \qquad k\times i=j \qquad k\times j=-i$$



$$\vec{A}\times\vec{B}=\left(A_yB_z-A_zB_y\right)i+\left(A_zB_x-A_xB_z\right)j+\left(A_xB_y-A_yB_x\right)k$$

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$$= (a_2 \times b_3 - a_3 \times b_2)\mathbf{i} - (a_1 \times b_3 - a_3 \times b_1)\mathbf{j} + (a_1 \times b_2 - a_2 \times b_1)$$

If $\vec{C} = \vec{A} \times \vec{B}$, the components of C are given by

$$C_x = A_y B_z - A_z B_y$$

$$C_y = A_z B_x - A_x B_z$$

$$C_z = A_x B_y - A_y B_x$$

Example 2

If $\vec{C} = \vec{A} \times \vec{B}$ **, where** $\vec{A} = 3i - 4j$ **, and** $\vec{B} = -2i + 3k$ **, what is** $\vec{C}$ **?**

Solution

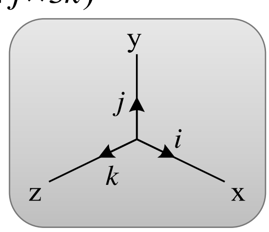
$$\vec{C} = \vec{A} \times \vec{B} = (3i - 4j) \times (-2i + 3k)$$

which, by distributive law, becomes

$$\vec{C} = -(3i \times 2i) + (3i \times 3k) + (4j \times 2i) - (4j \times 3k)$$

Using equation $\vec{A} \times \vec{B} = AB\sin\theta$ **to evaluate each term in the equation above we get**

$$\vec{C} = 0 - 9j - 8k - 12i = -12i - 9j - 8k$$



The vector C is perpendicular to both vectors A and B.

Example 3

Two vectors lying in the plane are given by the equations A = 2*i* + 3 *j* and B = -*i* + 2 *j* .

Find A×B and verify that A×B = -B× A

Solution

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix} i & j & k \\ 2 & 3 & 0 \\ -1 & 2 & 0 \end{vmatrix}=i\begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix}-j\begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix}+k\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}$$

$$\mathbf{A}\times\mathbf{B}=k\big(4-(-3)\big)=7k$$

$\mathbf{A} = 2i + 3\,j$ and $\mathbf{B} = -i + 2\,j$

$$\mathbf{B}\times\mathbf{A}=\begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ 2 & 3 & 0 \end{vmatrix}=i\begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix}-j\begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix}+k\begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix}$$

$$\mathbf{A}\times\mathbf{B}=k\big(-3-(4)\big)=-7k$$

So,

$$\mathbf{A}\times\mathbf{B} = \text{-}\mathbf{B}\times\mathbf{A}$$

Example 4

For the vectors $A = -3i + 7j - 4k$ and $B = 6i - 10j + 9k$, evaluate the expressions

$$\text{(a)}\ \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right)\ \text{and}\ \text{(b)}\ \sin^{-1}\left(\frac{|\mathbf{A}\times\mathbf{B}|}{|\mathbf{A}||\mathbf{B}|}\right)$$

$$|\mathbf{A}| = \sqrt{9+49+16} = \sqrt{74} \quad and \quad |\mathbf{B}| = \sqrt{36+100+81} = \sqrt{217}$$

$$|\mathbf{A}||\mathbf{B}| = \sqrt{74}\sqrt{217} = 126.720$$

$$\mathbf{A}\cdot\mathbf{B} = (-3i+7j-4k)\cdot(6i-10j+9k) = -18+(-70)+(-36) = -124$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} i & j & k \\ -3 & 7 & -4 \\ 6 & -10 & 9 \end{vmatrix} = i(63-40) - j(-27+24) + k(30-42)$$

$$\mathbf{A}\times\mathbf{B} = 23i + 3j - 12k \qquad |\mathbf{A}\times\mathbf{B}| = \sqrt{23^2+3^2+12^2} = 26.115$$

$$\cos^{-1}\left(\frac{-124}{126.720}\right) = \cos^{-1}(-0.9785) \cong 168^\circ$$

$$\sin^{-1}\left(\frac{26.115}{126.720}\right) = \sin^{-1}(0.0.2061) \cong 12^\circ$$